

XV. *On the Determination of the Exterior and Interior Attractions of Ellipsoids of Variable Densities.* By GEORGE GREEN, Esq., Caius College.

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THE determination of the attractions of ellipsoids, even on the hypothesis of a uniform density, has, on account of the utility and difficulty of the problem, engaged the attention of the greatest mathematicians. Its solution, first attempted by Newton, has been improved by the successive labours of Maclaurin, d'Alembert, Lagrange, Legendre, Laplace, and Ivory. Before presenting a new solution of such a problem, it will naturally be expected that I should explain in some degree the nature of the method to be employed for that end, in the following paper; and this explanation will be the more requisite, because, from a fear of encroaching too much upon the Society's time, some very comprehensive analytical theorems have been in the first instance given in all their generality.

It is well known, that when the attracted point  $p$  is situated within the ellipsoid, the solution of the problem is comparatively easy, but that from a breach of the law of continuity in the values of the attractions when  $p$  passes from the interior of the ellipsoid into the exterior space, the functions by which these attractions are given in the former case will not apply to the latter. As however this violation of the law of continuity may always be avoided by simply adding a positive quantity,  $u^2$  for instance, to that under the radical signs in the original integrals, it seemed probable that some advantage might thus be obtained, and the attractions in both cases, deduced from one common formula which would only require the auxiliary variable  $u$  to become evanescent in the final result. The principal advantage however which arises from the introduction of the new variable  $u$ , depends

on the property which a certain function  $V^*$  then possesses of satisfying a partial differential equation, whenever the law of the attraction is inversely as any power  $n$  of the distance. For by a proper application of this equation we may avoid all the difficulty usually presented by the integrations, and at the same time find the required attractions when the density  $\rho'$  is expressed by the product of two factors, one of which is a simple algebraic quantity, and the remaining one any rational and entire function of the rectangular co-ordinates of the element to which  $\rho'$  belongs.

The original problem being thus brought completely within the pale of analysis, is no longer confined as it were to the three dimensions of space. In fact,  $\rho'$  may represent a function of any number  $s$ , of independent variables, each of which may be marked with an accent, in order to distinguish this first system from another system of  $s$  analogous and unaccented variables, to be afterwards noticed, and  $V$  may represent the value of a multiple integral of  $s$  dimensions, of which every element is expressed by a fraction having for numerator the continued product of  $\rho'$  into the elements of all the accented variables, and for denominator a quantity containing the whole of these, with the unaccented ones also formed exactly on the model of the corresponding one in the value of  $V$  belonging to the original problem. Supposing now the auxiliary variable  $u$  is introduced, and the  $s$  integrations are effected, then will the resulting value of  $V$  be a function of  $u$  and of the  $s$  unaccented variable to be determined. But after the introduction

\* This function in its original form is given by

$$V = \int \frac{\rho' dx' dy' dz'}{\{(x' - x)^2 + (y' - y)^2 + (z' - z)^2\}^{\frac{n-1}{2}}},$$

where  $dx' dy' dz'$  represents the volume of any element of the attracting body of which  $\rho'$  is the density and  $x', y', z'$  are the rectangular co-ordinates;  $x, y, z$  being the co-ordinates of the attracted point  $p$ . But when we introduce the auxiliary variable  $u$  which is to be made equal to zero in the final result,

$$V = \int \frac{\rho' dx' dy' dz'}{\{(x' - x)^2 + (y' - y)^2 + (z' - z)^2 + u^2\}^{\frac{n-1}{2}}};$$

both integrals being supposed to extend over the whole volume of the attracting body.

of  $u$ , the function  $V$  has the property of satisfying a partial differential equation of the second order, and by an application of the Calculus of Variations it will be proved in the sequel that the required value of  $V$  may always be obtained by merely satisfying this equation, and certain other simple conditions when  $\rho'$  is equal to the product of two factors, one of which may be any rational and entire function of the  $s$  accented variables, the remaining one being a simple algebraic function whose form continues unchanged, whatever that of the first factor may be.

The chief object of the present paper is to resolve the problem in the more extended signification which we have endeavoured to explain in the preceding paragraph, and, as is by no means unusual, the simplicity of the conclusions corresponds with the generality of the method employed in obtaining them. For when we introduce other variables connected with the original ones by the most simple relations, the rational and entire factor in  $\rho'$  still remains rational and entire of the same degree, and may under its altered form be expanded in a series of a finite number of similar quantities, to each of which there corresponds a term in  $V$ , expressed by the product of two factors; the first being a rational and entire function of  $s$  of the new variables entering into  $V$ , and the second a function of the remaining new variable  $h$ , whose differential coefficient is an algebraic quantity. Moreover the first is immediately deducible from the corresponding part of  $\rho'$  without calculation.

The solution of the problem in its extended signification being thus completed, no difficulties can arise in applying it to particular cases. We have therefore on the present occasion given two applications only. In the first, which relates to the attractions of ellipsoids, both the interior and exterior ones are comprised in a common formula agreeably to a preceding observation, and the discontinuity before noticed falls upon one of the independent variables, in functions of which both these attractions are expressed; this variable being constantly equal to zero so long as the attracted point  $p$  remains within the ellipsoid, but becoming equal to a determinate function of the co-

ordinates of  $p$ , when  $p$  is situated in the exterior space. Instead too of seeking directly the value of  $V$ , all its differentials have first been deduced, and thence the value of  $V$  obtained by integration. This slight modification has been given to our method, both because it renders the determination of  $V$  in the case considered more easy, and may likewise be usefully employed in the more general one before mentioned. The other application is remarkable both on account of the simplicity of the results to which it leads, and of their analogy with those obtained by Laplace. (*Méc. Cél. Liv. III. Chap. 2.*) In fact, it would be easy to shew that these last are only particular cases of the more general ones contained in the article now under notice.

The general solution of the partial differential equation of the second order, deducible from the seventh and three following articles of this paper, and in which the principal variable  $V$  is a function of  $s + 1$  independent variables, is capable of being applied with advantage to various interesting physico-mathematical enquiries. Indeed the law of the distribution of heat in a body of ellipsoidal figure, and that of the motion of a non-elastic fluid over a solid obstacle of similar form, may be thence almost immediately deduced; but the length of our paper entirely precludes any thing more than an allusion to these applications on the present occasion.

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1. The object of the present paper will be to exhibit certain general analytical formulæ, from which may be deduced as a very particular case the values of the attractions exerted by ellipsoids upon any exterior or interior point, supposing their densities to be represented by functions of great generality.

Let us therefore begin with considering  $\rho'$  as a function of the  $s$  independent variables

$$x'_1, \quad x'_2, \quad x'_3 \dots \dots \dots x'_s,$$

and let us afterwards form the function

$$V = \int \frac{dx'_1 dx'_2 dx'_3 \dots \dots \dots dx'_s \cdot \rho'}{\{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + \dots \dots \dots + (x_s - x'_s)^2 + u^2\}^{\frac{n-1}{2}}} \dots \dots \dots (1)$$

the sign  $\int$  serving to indicate  $s$  integrations relative to the variables  $x_1', x_2', x_3', \dots, x_s'$ , and similar to the double and triple ones employed in the solution of geometrical and mechanical problems. Then it is easy to perceive that the function  $V$  will satisfy the partial differential equation

$$0 = \frac{d^2 V}{dx_1^2} + \frac{d^2 V}{dx_2^2} + \dots + \frac{d^2 V}{dx_s^2} + \frac{d^2 V}{du^2} + \frac{n-s}{u} \frac{dV}{du} \dots \dots \dots (2)$$

seeing that in consequence of the denominator of the expression (1), every one of its elements satisfies for  $V$  to the equation (2).

To give an example of the manner in which the multiple integral is to be taken, we may conceive it to comprise all the real values both positive and negative of the variables  $x_1', x_2', \dots, x_s'$ , which satisfy the condition

$$\frac{x_1'^2}{a_1'^2} + \frac{x_2'^2}{a_2'^2} + \frac{x_3'^2}{a_3'^2} + \dots + \frac{x_s'^2}{a_s'^2} < 1 \dots \dots \dots (a)$$

the symbol  $<$ , as is the case also in what follows, not excluding equality.

2. In order to avoid the difficulties usually attendant on integrations like those of the formula (1), it will here be convenient to notice two or three very simple properties of the function  $V$ .

In the first place, then, it is clear that the denominator of the formula (1) may always be expanded in an ascending series of the entire powers of the increments of the variables  $x_1, x_2, \dots, x_s, u$ , and their various products by means of Taylor's Theorem, unless we have simultaneously

$$x_1 = x_1', \quad x_2 = x_2', \dots, x_s = x_s' \quad \text{and} \quad u = 0;$$

and therefore  $V$  may always be expanded in a series of like form, unless the  $s+1$  equations immediately preceding are all satisfied for one at least of the elements of  $V$ . It is thus evident that the function  $V$  possesses the property in question, except only when the two conditions

$$\frac{x_1^2}{a_1'^2} + \frac{x_2^2}{a_2'^2} + \frac{x_3^2}{a_3'^2} + \dots + \frac{x_s^2}{a_s'^2} < 1 \text{ and } u = 0 \dots \dots (3)$$

are satisfied simultaneously, considering as we shall in what follows the limits of the multiple integral (1) to be determined by the condition (a)\*.

In like manner it is clear that when

$$\frac{x_1^2}{a_1'^2} + \frac{x_2^2}{a_2'^2} + \dots + \frac{x_s^2}{a_s'^2} > 1 \dots \dots (4),$$

the expansion of  $V$  in powers of  $u$  will contain none but the even powers of this variable.

Again, it is quite evident from the form of the function  $V$  that when any one of the  $s + 1$  independent variables therein contained becomes infinite, this function will vanish of itself.

3. The three foregoing properties of  $V$  combined with the equation (2) will furnish some useful results. In fact, let us consider the quantity

$$\int dx_1 dx_2 \dots dx_s du u^{n-s} \cdot \left\{ \left( \frac{dV}{dx_1} \right)^2 + \left( \frac{dV}{dx_2} \right)^2 + \dots + \left( \frac{dV}{dx_s} \right)^2 + \left( \frac{dV}{du} \right)^2 \right\} \dots \dots (5)$$

where the multiple integral comprises all the real values whether positive or negative of  $x_1, x_2, \dots, x_s$ , with all the real and positive values of  $u$  which satisfy the condition

$$\frac{x_1^2}{a_1'^2} + \frac{x_2^2}{a_2'^2} + \dots + \frac{x_s^2}{a_s'^2} + \frac{u^2}{h^2} < 1 \dots \dots (6)$$

\* The necessity of this first property does not explicitly appear in what follows, but it must be understood in order to place the application of the method of integration by parts, in Nos. 3, 4, and 5, beyond the reach of objection. In fact, when  $V$  possesses this property, the theorems demonstrated in these Nos. are certainly correct: but they are not necessarily so for every form of the function  $V$ , as will be evident from what has been shewn in the third article of my Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.

$a_1, a_2, \dots, a_s$  and  $h$  being positive constant quantities; and such that we may have generally

$$a_r > a'_r.$$

In this case the multiple integral (5) will have two extreme limits, viz. one in which the conditions

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \dots + \frac{x_s^2}{a_s^2} + \frac{u^2}{h^2} = 1 \text{ and } u = \text{a positive quantity} \dots \dots (7)$$

are satisfied; and another defined by

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \dots + \frac{x_s^2}{a_s^2} < 1 \text{ and } u = 0.$$

Moreover, for greater distinctness, we shall mark the quantities belonging to the former with two accents, and those belonging to the latter with one only.

Let us now suppose that  $V''$  is completely given, and likewise  $V'_1$  or that portion of  $V'$  in which the condition (3) is satisfied; then if we regard  $V'_2$  or the rest of  $V'$  as quite arbitrary, and afterwards endeavour to make the quantity (5) a minimum, we shall get in the usual way, by applying the Calculus of Variations,

$$0 = - \int dx_1 dx_2 \dots dx_s du u^{n-s} \delta V \left\{ \sum_1^{s+1} \frac{d^2 V}{dx_r^2} + \frac{d^2 V}{du^2} + \frac{n-s}{u} \frac{dV}{du} \right\} \\ - \int dx_1 dx_2 \dots dx_s u'^{n-s} \delta V'_2 \frac{dV'_2}{du} \dots \dots (8)$$

seeing that  $\delta V'' = 0$  and  $\delta V'_1 = 0$ , because the quantities  $V''$  and  $V'_1$  are supposed given.

The first line of the expression immediately preceding gives generally

$$0 = \sum_1^{s+1} \frac{d^2 V}{dx_r^2} + \frac{d^2 V}{du^2} + \frac{n-s}{u} \frac{dV}{du} \dots \dots (2')$$

which is identical with the equation (2) No. 1, and the second line gives

$$0 = u'^{n-s} \frac{dV'_2}{du} (u' \text{ being evanescent}) \dots \dots (9).$$

From the nature of the question *de minimo* just resolved, there can be little doubt but that the equations (2') and (9) will suffice for the complete determination of  $V$ , where  $V''$  and  $V_1'$  are both given. But as the truth of this will be of consequence in what follows, we will, before proceeding farther, give a demonstration of it; and the more willingly because it is simple and very general.

4. Now since in the expression (5)  $u$  is always positive, every one of the elements of this expression will therefore be positive; and as moreover  $V''$  and  $V_1'$  are given, there must necessarily exist a function  $V_0$  which will render the quantity (5) a proper minimum. But it follows, from the principles of the Calculus of Variations, that this function  $V_0$ , whatever it may be, must moreover satisfy the equations (2') and (9). If then there exists any other function  $V_1$  which satisfies the last-named equations, and the given values of  $V''$  and  $V_1'$ , it is easy to perceive that the function

$$V = V_0 + A(V_1 - V_0)$$

will do so likewise, whatever the value of the arbitrary constant quantity  $A$  may be. Suppose therefore that  $A$  originally equal to zero is augmented successively by the infinitely small increments  $\delta A$ , then the corresponding increment of  $V$  will be

$$\delta V = (V_1 - V_0) \delta A,$$

and the quantity (5) will remain constantly equal to its minimum value, however great  $A$  may become, seeing that by what precedes the variation of this quantity must be equal to zero whatever the variation of  $V$  may be, provided the foregoing conditions are all satisfied. If then, besides  $V_0$  there exists another function  $V_1$  satisfying them all, we might give to the partial differentials of  $V$ , any values however great, by augmenting the quantity  $A$  sufficiently, and thus cause the quantity (5) to exceed any finite positive one, contrary to what has just been proved. Hence no such value as  $V_1$  exists.

We thus see that when  $V''$  and  $V_1'$  are both given, there is one and only one way of satisfying simultaneously the partial differential equation (2), and the condition (9).

5. Again, it is clear that the condition (4) is satisfied for the whole of  $V'_2$ ; and it has before been observed (No. 2.) that when  $V$  is determined by the formula (1), it may always be expanded in a series of the form

$$V = A + Bu^2 + Cu^4 + \&c.$$

Hence the right side of the equation (9) is a quantity of the order  $u'^{n-s+1}$ ; and  $u'$  being evanescent, this equation will then evidently be satisfied, provided we suppose, as we shall in what follows, that

$$n - s + 1 \text{ is positive.}$$

If now we could by any means determine the values of  $V''$  and  $V'_1$  belonging to the expression (1), the value of  $V$  would be had without integration by simply satisfying (2') and (9), as is evident from what precedes. But by supposing all the constant quantities  $a_1, a_2, a_3, \dots, a_s$ , and  $h$  infinite, it is clear that we shall have

$$0 = V'',$$

and then we have only to find  $V'_1$ , and thence deduce the general value of  $V$ .

6. For this purpose let us consider the quantity

$$\int dx_1 dx_2 \dots dx_s du u^{n-s} \left\{ \frac{dV}{dx_1} \frac{dU}{dx_1} + \frac{dV}{dx_2} \frac{dU}{dx_2} + \dots + \frac{dV}{dx_s} \frac{dU}{dx_s} + \frac{dV}{du} \frac{dU}{du} \right\}; \dots \dots (10)$$

the limits of the multiple integral being the same as those of the expression (5), and  $U$  being a function of  $x_1, x_2, \dots, x_s$  and  $u$ , satisfying the condition  $0 = U''$  when  $a_1, a_2, \dots, a_s$  and  $h$  are infinite.

But the method of integration by parts reduces the quantity (10) to

$$\begin{aligned} & - \int dx_1 dx_2 \dots dx_s \frac{dU'}{du} u'^{n-s} \cdot V' \\ & - \int dx_1 dx_2 \dots dx_s du u^{n-s} V \left\{ \sum_{i=1}^{s+1} \frac{d^2 U}{dx_i^2} + \frac{d^2 U}{du^2} + \frac{n-s}{u} \frac{dU}{du} \right\} \dots \dots \dots (11) \end{aligned}$$

since  $0 = V''$ ; and as we have likewise  $0 = U''$ , the same quantity (10) may also be put under the form

$$\begin{aligned}
& - \int dx_1 dx_2 \dots dx_s \frac{dV'}{du} u'^{n-s} \cdot U' \\
& - \int dx_1 dx_2 \dots dx_s du u'^{n-s} \cdot U \left\{ \sum_1^{s+1} \frac{d^2 V}{dx_r^2} + \frac{d^2 V}{du^2} + \frac{n-s}{u} \frac{dV}{du} \right\} \dots \dots (12).
\end{aligned}$$

Supposing therefore that  $U$  like  $V$  also satisfies the equation (2'), each of the expressions (11) and (12) will be reduced to its upper line, and we shall get by equating these two forms of the same quantity:

$$\int dx_1 dx_2 \dots dx_s \frac{dU'}{du} u'^{n-s} V' = \int dx_1 dx_2 \dots dx_s \frac{dV'}{du} u'^{n-s} U';$$

the quantities bearing an accent belonging, as was before explained, to one of the extreme limits.

Because  $V$  satisfies the condition (9), the equation immediately preceding may be written

$$\int dx_1 dx_2 \dots dx_s \frac{dU'}{du} u'^{n-s} V' = \int dx_1 dx_2 \dots dx_s \frac{dV'_1}{du} u'^{n-s} U'_1.$$

If now we give to the general function  $U$  the particular value

$$U = \{(x_1 - x_1'')^2 + (x_2 - x_2'')^2 + \dots + (x_s - x_s'')^2 + u^2\}^{\frac{1-n}{2}};$$

which is admissible, since it satisfies for  $V$  to the equation (2), and gives  $U'' = 0$ , the last formula will become

$$\begin{aligned}
& \int \frac{dx_1 dx_2 \dots dx_s u'^{n-s} \frac{dV'_1}{du}}{\{(x_1 - x_1'')^2 + (x_2 - x_2'')^2 + \dots + (x_s - x_s'')^2 + u'^2\}^{\frac{n-1}{2}}} \\
& = \int \frac{dx_1 dx_2 \dots dx_s \cdot (1-n) u'^{n-s+1} V'}{\{(x_1 - x_1'')^2 + (x_2 - x_2'')^2 + \dots + (x_s - x_s'')^2 + u'^2\}^{\frac{n+1}{2}}} \dots \dots (13);
\end{aligned}$$

in which expression  $u'$  must be regarded as an evanescent positive quantity.

In order now to effect the integrations indicated in the second member of this equation, let us make

$x_1 - x_1'' = u' \rho \cos \theta_1$ ;  $x_2 - x_2'' = u' \rho \sin \theta_1 \cos \theta_2$ ;  $x_3 - x_3'' = u' \rho \sin \theta_1 \sin \theta_2 \cos \theta_3$ , &c.  
 until we arrive at the two last, viz.,

$$x_{s-1} - x_{s-1}'' = u' \rho \sin \theta_1 \sin \theta_2 \dots \sin \theta_{s-2} \cos \theta_{s-1},$$

$$x_s - x_s'' = u' \rho \sin \theta_1 \sin \theta_2 \dots \sin \theta_{s-2} \sin \theta_{s-1};$$

$u'$  being, as before, a vanishing quantity.

Then by the ordinary formulæ for the transformation of multiple integrals we get

$$dx_1 dx_2 \dots dx_s = u'^s \rho^{s-1} \sin \theta_1^{s-2} \sin \theta_2^{s-3} \dots \sin \theta_{s-2} d\rho d\theta_1 d\theta_2 \dots d\theta_{s-1},$$

and the second number of the equation (13) by substitution will become

$$\int \frac{d\rho d\theta_1 d\theta_2 \dots d\theta_{s-1} \rho^{s-1} \sin \theta_1^{s-2} \sin \theta_2^{s-3} \dots \sin \theta_{s-2} \cdot (1-n)V'}{(1+\rho^2)^{\frac{n+1}{2}}} \dots (14).$$

But since  $u'$  is evanescent, we shall have  $\rho$  infinite, whenever  $x_1, x_2, \dots x_s$  differ sensibly from  $x_1'', x_2'', \dots x_s''$ ; and as moreover  $n-s+1$  is positive, it is easy to perceive that we may neglect all the parts of the last integral for which these differences are sensible. Hence  $V'$  may be replaced with the constant value  $V'_0$  in which we have generally

$$x_r = x_r''.$$

Again, because the integrals in (14) ought to be taken from  $\theta_{s-1}=0$  to  $\theta_{s-1}=2\pi$ , and afterwards from  $\theta_r=0$  to  $\theta_r=\pi$ , whatever whole number less than  $s-1$  may be represented by  $r$ , we easily obtain by means of the well known function Gamma:

$$\int \sin \theta_1^{s-2} \sin \theta_2^{s-3} \sin \theta_3^{s-4} \dots \sin \theta_{s-2} d\theta_1 d\theta_2 \dots d\theta_{s-1} = \frac{2\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)};$$

and as by the aid of the same function we readily get

$$\int_0^\infty \frac{\rho^{s-1} d\rho}{(1+\rho^2)^{\frac{n+1}{2}}} = \frac{\Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{n-s+1}{2}\right)}{2\Gamma\left(\frac{n+1}{2}\right)},$$

the integral (14) will in consequence become

$$\frac{-2\pi^{\frac{s}{2}} \cdot \Gamma\left(\frac{n-s+1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} V'_0,$$

and thus the equation (13) will take the form

$$\int \frac{dx_1 dx_2 \dots dx_s u'^{n-s} \frac{dV'_1}{du}}{\{(x_1 - x_1'')^2 + (x_2 - x_2'')^2 + \dots + (x_s - x_s'')^2 + u'^2\}^{\frac{n-1}{2}}} = \frac{-2\pi^{\frac{s}{2}} \cdot \Gamma\left(\frac{n-s+1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} V'_0.$$

In this equation  $V$  is supposed to be such a function of  $x_1, x_2, \dots, x_s$  and  $u$ , that the equation (2) and condition (9) are both satisfied. Moreover  $V''=0$ , and  $V'_0$  is the particular value of  $V$  for which

$$x_1 = x_1''; \quad x_2 = x_2''; \dots x_s = x_s'', \text{ and } u = 0.$$

Let us now make, for abridgment,

$$P = u^{n-s} \frac{dV}{du}, \text{ (when } u=0) \dots \dots \dots (b),$$

and afterwards change  $x$  into  $x'$ , and  $x''$  into  $x$  in the expression immediately preceding, there will then result

$$\int \frac{dx'_1 dx'_2 \dots dx'_s P'_1}{\{(x'_1 - x_1)^2 + (x'_2 - x_2)^2 + \dots + (x'_s - x_s)^2 + u'^2\}^{\frac{n-1}{2}}} = \frac{-2\pi^{\frac{s}{2}} \Gamma\left(\frac{n-s+1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} V' \dots (15),$$

$P'$  being what  $P$  becomes by changing generally  $x_r$  into  $x'_r$ , the unit attached to the foot of  $P'$  indicating, as before, that the multiple integral comprises only the values admitted by the condition (a), and  $V'$  being what  $V$  becomes when we make  $u = 0$ .

The equation just given supposes  $u'$  evanescent; but if we were to replace  $u'$  with the general value  $u$  in the first member, and make a corresponding change in the second by replacing  $V'$  with the general value  $V$ , this equation would still be correct, and we should thus have

$$\int \frac{dx_1' dx_2' \dots dx_s' P_1'}{\{(x_1' - x_1)^2 + (x_2' - x_2)^2 + \dots + (x_s' - x_s)^2 + u^2\}^{\frac{n-1}{2}}} = \frac{-2\pi^{\frac{s}{2}} \Gamma\left(\frac{n-s+1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} V \dots (16).$$

For under the present form both its members evidently satisfy the equation (2), the condition (9), and give  $V'' = 0$ . Moreover, when the condition (3) is satisfied, the same members are equal in consequence of (15). Hence by what has before been proved (No. 4), they are necessarily equal in general.

By comparing the equation (16) with the formula (1), it will become evident, that whenever we can by any means obtain a value of  $V$  satisfying the foregoing conditions, we shall always be able to assign a value of  $\rho'$  which substituted in (1) shall reproduce this value of  $V$ . In fact, by omitting the unit at the foot of  $P'$ , which only serves to indicate the limits of the integral, we readily see that the required value of  $\rho'$  is

$$\rho' = - \frac{\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{s}{2}} \Gamma\left(\frac{n-s+1}{2}\right)} P' \dots \dots \dots (c).$$

7. The foregoing results being obtained, it will now be convenient to introduce other independent variables in the place of the original ones, such that

$$x_1 = a_1 \xi_1, \quad x_2 = a_2 \xi_2, \dots \dots \dots x_s = a_s \xi_s, \quad u = h\nu,$$

$a_1, a_2, \dots \dots a_s$  being functions of  $h$ , one of the new independent variables, determined by

$$a_1'^2 = a_1'^2 + h^2, \quad a_2'^2 = a_2'^2 + h^2, \dots \dots \dots a_s'^2 = a_s'^2 + h^2,$$

and  $\nu$  a function of the remaining new variables,  $\xi_1, \xi_2, \xi_3, \dots \dots \xi_s$ , satisfying the equation

$$1 = \nu^2 + \xi_1^2 + \xi_2^2 + \dots \dots \dots + \xi_s^2;$$

$a_1', a_2', a_3', \dots \dots a_s'$  being the same constant quantities as in the equation (a), No 1. Moreover,  $a_1, a_2, \dots \dots a_s$  will take the values belonging to the extreme limit before marked with two accents, by simply assigning to  $h$  an infinite value.

The easiest way of transforming the equation (2) will be to remark, that it is the general one which presents itself when we apply the Calculus of Variations to the quantity (5), in order to render it a minimum. We have therefore in the first place

$$\left(\frac{dV}{du}\right)^2 + \sum_1^{s+1} \left(\frac{dV}{dx_r}\right)^2 = \sum_1^{s+1} \left(\frac{dV}{a_r d\xi_r}\right)^2 + \left\{ \left(\frac{dV}{dh}\right)^2 - \left(\sum_1^{s+1} \frac{h\xi_r}{a_r^2} \frac{dV}{d\xi_r}\right)^2 \right\} \left(1 - \sum_1^{s+1} \frac{a_r'^2 \xi_r^2}{a_r^2}\right)^{-1};$$

and by the ordinary formula for the transformation of multiple integrals,

$$dx_1 dx_2 \dots dx_s du = \frac{a_1 a_2 \dots a_s}{v} \left(1 - \sum_1^{s+1} \frac{a_r'^2 \xi_r^2}{a_r^2}\right) d\xi_1 d\xi_2 \dots d\xi_s dh.$$

$$\text{But since } 1 - \sum_1^{s+1} \frac{a_r'^2 \xi_r^2}{a_r^2} = v^2 + h^2 \sum_1^{s+1} \frac{\xi_r^2}{a_r^2},$$

the expression (5) after substitution will become

$$f d\xi_1 d\xi_2 \dots d\xi_s dh a_1 a_2 a_3 \dots a_s h^{n-s} v^{n-s-1} \dots \left\{ \left(v^2 + h^2 \sum_1^{s+1} \frac{\xi_r^2}{a_r^2}\right) \sum_1^{s+1} \left(\frac{dV}{a_r d\xi_r}\right)^2 + \left(\frac{dV}{dh}\right)^2 - h^2 \left(\sum_1^{s+1} \frac{\xi_r}{a_r^2} \frac{dV}{d\xi_r}\right)^2 \right\}.$$

Applying now the method of integration by parts to the variation of this quantity, by reduction, we get for the equivalent of (2) the equation

$$\begin{aligned} 0 &= \frac{d^2 V}{dh^2} + \left(n - \sum \frac{a_r'^2}{a_r^2}\right) \frac{dV}{h dh} + (1 - \sum \xi_r^2) \sum \frac{d^2 V}{a_r^2 d\xi_r^2} + (s - n - 1) \sum \frac{\xi_r}{a_r^2} \frac{dV}{d\xi_r}, \\ &+ h^2 \sum \frac{\xi_r^2}{a_r^2} \times \sum \frac{d^2 V}{a_r^2 d\xi_r^2} - h^2 \sum \sum \frac{\xi_r \xi_{r'}}{a_r^2 a_{r'}^2} \frac{d^2 V}{d\xi_r d\xi_{r'}} \dots \dots \dots (2'') \\ &+ h^2 \sum \frac{\xi_r dV}{a_r^4 d\xi_r} - h^2 \sum \frac{1}{a_r^2} \times \sum \frac{\xi_r dV}{a_r^2 d\xi_r}; \end{aligned}$$

where the finite integrals are all supposed taken from  $r = 1$  to  $r = s + 1$ , and from  $r' = 1$  to  $r' = s + 1$ .

The last equation may be put under the abridged form,

$$0 = \frac{d^2 V}{dh^2} + \left(n - \sum \frac{a_r'^2}{a_r^2}\right) \frac{dV}{h dh} + \nabla V \dots \dots \dots (2'''),$$

provided we have generally

$$\text{coefficient of } \frac{d^2 V}{d\xi_r^2} \text{ in } \nabla V = \frac{1}{a_r^2} \{1 - \xi_r^2 - \sum_{1}^{s+1} \frac{a_r'^2}{a_r^2} \xi_r'^2 + \frac{a_r'^2}{a_r^2} \xi_r'^2\},$$

$$\text{coefficient of } \frac{d^2 V}{d\xi_r d\xi_r'} \text{ in } \nabla V = -\frac{2h^2}{a_r^2 a_r'^2} \xi_r \xi_r',$$

$$\text{coefficient of } \frac{dV}{d\xi_r} \text{ in } \nabla V = \frac{\xi_r}{a_r^2} \left\{ -n + \sum \frac{a_r'^2}{a_r^2} - \frac{a_r'^2}{a_r^2} \right\}.$$

Moreover, when we employ the new variables

$$\frac{dV}{du} = -v \left( 1 - \sum \frac{a_r'^2 \xi_r'^2}{a_r^2} \right)^{-1} \cdot \left\{ \sum \frac{h \xi_r}{a_r^2} \frac{dV}{d\xi_r} - \frac{dV}{dh} \right\},$$

and therefore the condition (9) in like manner will become

$$0 = v^{n-s+1} h^{n-s} \left( 1 - \sum \frac{a_r'^2 \xi_r'^2}{a_r^2} \right)^{-1} \cdot \left\{ \sum \frac{h \xi_r}{a_r^2} \frac{dV}{d\xi_r} - \frac{dV}{dh} \right\} \dots \dots \dots (9');$$

where the values of the variables  $\xi_1, \xi_2, \dots, \xi_s$  must be such as satisfy the equation  $v^2=0$ , whatever  $h$  may be; and as  $n-s+1$  is positive, it is clear that this condition will always be satisfied, provided the partial differentials of  $V$  relative to the new variables are all finite.

8. Let us now try whether it is possible to satisfy the equation (2''') by means of a function of the form

$$V = H\phi \dots \dots \dots (\beta);$$

$H$  depending on the variable  $h$  only, and  $\phi$  being a rational and entire function of  $\xi_1, \xi_2, \dots, \xi_s$  of the degree  $\gamma$ , and quite independent of  $h$ .

By substituting this value of  $V$  in (2''') and making

$$0 = \frac{d^2 H}{dh^2} + \left( n - \sum \frac{a_r'^2}{a_r^2} \right) \frac{dH}{h dh} + \kappa H \dots \dots \dots (17),$$

we readily get

$$0 = \nabla \phi - \kappa \phi \dots \dots \dots (18);$$

where, in virtue of (17)  $\kappa$  must necessarily be a function of  $h$  only; and as the required value of  $\phi$ , if it exist, must be independent of  $h$ , we have, by making  $h=0$  in the equation immediately preceding,

$$0 = \nabla' \phi - k_0 \phi \dots \dots \dots (19);$$

$k_0$  being the value  $\kappa$ , and  $\nabla' \phi$  that of  $\nabla \phi$  when  $h=0$ .

We shall demonstrate almost immediately that every function  $\phi$  of the form (20), No. 9, which satisfies the equation (19), and which therefore is independent of  $h$ , will likewise satisfy the equation (18); and the corresponding value of  $\kappa$  obtained from the latter being substituted in the ordinary differential equation (17), we shall only have to integrate this last in order to have a proper value of  $V$ .

9. To satisfy the equation (19) let us assume

$$\phi = F(\xi_1^2, \xi_2^2, \xi_3^2, \dots, \xi_s^2) \xi_p, \xi_q, \&c. \dots \dots \dots (20);$$

$F$  being the characteristic of a rational and entire function of the degree  $2\gamma'$ , and the most general of its kind, and  $\xi_p, \xi_q, \&c.$  designating the variables in  $\phi$  which are affected with odd exponents only; so that if their number be  $\nu$  we shall have

$$\gamma = 2\gamma' + \nu,$$

the remaining variables having none but even exponents. Then it is easy to perceive, that after substitution the second member of the equation (19) will be precisely of the same form as the assumed value of  $\phi$ , and by equating separately to zero the coefficients of the various powers and products of  $\xi_1, \xi_2, \dots, \xi_s$ , we shall obtain just the same number of linear algebraic equations as there are coefficients in  $\phi$ , and consequently be enabled to determine the ratios of these coefficients together with the constant quantity  $k_0$ .

In fact, by writing the foregoing value of  $\phi$  under the form

$$\phi = SA_{m_1, m_2, \dots, m_s} \xi_1^{m_1} \xi_2^{m_2} \dots \xi_s^{m_s} \dots \dots \dots (20');$$

and proceeding as above described, the coefficient of  $\xi_1^{m_1} \xi_2^{m_2} \dots \xi_s^{m_s}$  will give the general equation

$$\begin{aligned} 0 = & \left\{ \sum_1^{s+1} \frac{m_r(m_r - s + n)}{a_r'^2} + k_0 \right\} A_{m_1, m_2, \dots, m_s} \\ & + \sum \sum \frac{(m_r + 2)(m_r + 1)}{a_r'^2} A_{m_1, m_2, \dots, m_r+2, \dots, m_r'-2, \dots, m_s, \dots \dots \dots} \\ & - \sum_1^{s+1} \frac{(m_r + 2)(m_r + 1)}{a_r'^2} A_{m_1, m_2, \dots, m_r+2, \dots, m_s}; \end{aligned} \quad (21)$$

the double finite integral comprising all the values of  $r$  and  $r'$ , except those in which  $r = r'$ , and consequently containing when completely expanded  $s(s-1)$  terms.

For the terms of the highest degree  $\gamma$  and of which the number is

$$\frac{\gamma' + 1 . \gamma' + 2 . \dots \gamma' + s - 1}{1 . 2 . 3 . \dots s - 1} = N,$$

the last line of the expression (21) evidently vanishes, and thus we obtain  $N$  distinct linear equations between the coefficients of the degree  $\gamma$  in  $\phi$  and  $k_0$ .

Moreover, from the form of these equations it is evident that we may obtain by elimination one equation in  $k_0$  of the degree  $N$ , of which each of the  $N$  roots will give a distinct value of the function  $\phi^{(\gamma)}$ , having one arbitrary constant for factor; the homogeneous function  $\phi^{(\gamma)}$  being composed of all the terms of the highest degree,  $\gamma$  in  $\phi$ . But the coefficients of  $\phi^{(\gamma)}$  and  $k_0$  being known, we may thence easily deduce all the remaining coefficients in  $\phi$ , by means of the formula (21).

Now, since the  $N$  linear equations have no terms except those of which the coefficients of  $\phi^{(\gamma)}$  are factors, it follows that if  $k_0$  were taken at will, the resulting values of all these coefficients would be equal to zero. If however we obtain the values of  $N-1$  of the coefficients in terms of the remaining one  $A$  from  $N-1$  of the equations, by the ordinary formulæ, and substitute these in the remaining equation, we shall get a result of the form

$$K . A = 0,$$

where  $K$  is a function of  $k_0$  of the degree  $N$ . We shall thus have only two cases to consider: First, that in which  $A=0$ , and consequently also all the other coefficients of  $\phi^{(\gamma)}$  together with the remaining ones in  $\phi$ , as will be evident from the formulæ (21). Hence, in this case

$$\phi = 0:$$

Secondly, that in which  $k_0$  is one of the  $N$  roots of  $0=K$ , as for instance,  $k'_0$  in this case all the coefficients of  $\phi$  will become multiples of  $A$ ; and we shall have

$$\phi = A \phi_1:$$

$\phi_1$  being a determinate function of  $\xi_1, \xi_2, \dots, \xi_s$ .

We thus see that when we consider functions of the form (20) only, the most general solution that the equation

$$0 = \nabla' \bar{\phi} - k'_0 \bar{\phi} \dots \dots \dots (19')$$

admits is

$$\dots \dots \dots \text{or, } \bar{\phi} = 0; \quad \text{or, } \bar{\phi} = \alpha \phi;$$

$\alpha$  being a quantity independent of  $\xi_1, \xi_2, \dots, \xi_s$ , and  $\phi$  any function which satisfies for  $\bar{\phi}$  to the equation (19'). But by affecting both sides of the equation

$$0 = \nabla' \phi - k'_0 \phi$$

with the symbol  $\nabla$ , we get

$$0 = \nabla \cdot \nabla' \phi - k'_0 \cdot \nabla \phi;$$

and we shall afterwards prove the operations indicated by  $\nabla$  and  $\nabla'$  to be such, that whatever  $\phi$  may be,

$$\nabla \nabla' \phi = \nabla' \nabla \phi.$$

Hence, the last equation becomes

$$\nabla' (\nabla \phi) - k'_0 \nabla \phi;$$

and as  $\nabla \phi$  like  $\phi$  is of the form (20), it follows from what has just been shewn, that

$$\text{either } 0 = \nabla \phi, \quad \text{or, } \nabla \phi = \alpha \phi,$$

$\alpha$  being a quantity independent of  $\xi_1, \xi_2, \dots, \xi_s$ .

The first is inadmissible, since it would give  $\phi = 0$ ; therefore when  $\phi$  satisfies (19'), we have

$$\nabla \phi' = \alpha \phi, \quad \text{i. e. } 0 = \nabla \phi - \alpha \phi.$$

But since  $\alpha$  is independent of  $\xi_1, \xi_2, \dots, \xi_s$ , this last equation is evidently identical with (18), since the equation (18) merely requires that  $\kappa$  should be independent of  $\xi_1, \xi_2, \dots, \xi_s$ .

Having thus proved that every function of the form (20) which satisfies (19) will likewise satisfy (18), it will be more simple to determine the remaining coefficients of  $\phi$  from those of  $\phi^{(\gamma)}$  by means of the last equation, than to employ the formula (21) for that purpose.

Making therefore  $h$  infinite in (18), and writing  $\frac{k_1}{h^2}$  in the place of  $\kappa$ , we get

$$0 = \Sigma_1^{s+1} \cdot (1 - \xi_r^2) \frac{d^2 \phi}{d\xi_r^2} - 2(\Sigma\Sigma) \xi_r \xi_{r'} \frac{d^2 \phi}{d\xi_r d\xi_{r'}} - n \Sigma_1^{s+1} \xi_r \frac{d\phi}{d\xi_r} - k_1 \phi;$$

where  $(\Sigma\Sigma)$  comprises the  $\frac{s(s-1)}{1 \cdot 2}$  combinations which can be formed of the  $s$  indices taken in pairs.

If now we substitute the value of  $\phi$  before given (20'), and recollect that for the terms of the highest degree we have  $\Sigma m_r = \gamma$ , we shall readily get

$$0 = (\gamma - \Sigma m_r) (\gamma + \Sigma m_r + n - 1) A_{m_1, m_2, \dots, m_s} + (m_r + 2) (m_r + 1) A_{m_1, m_r+2, \dots, m_s} \dots (22),$$

from which all the remaining coefficients in  $\phi$  will readily be deduced, when those of the part  $\phi^{(\gamma)}$  are known.

10. It now remains, as was before observed, to integrate the ordinary differential equation (17) No. 8. But, by the known theory of linear equations, the integration of (17) will always become more simple when we have a particular value satisfying it, and fortunately in the present case such a value may always be obtained from  $\phi$  by simply changing  $\xi_r$  into  $\frac{a_r}{\sqrt{(\Sigma a_r'^2)}}$ . In fact if we represent the value thus obtained by  $H_0$  we shall have

$$\frac{dH_0}{dh} = \Sigma_1^{s+1} \frac{d\phi}{d\xi_r} \cdot \frac{h}{a_r \sqrt{(\Sigma a_r'^2)}},$$

and by a second differentiation

$$\frac{d^2 H_0}{dh^2} = \Sigma \frac{d\phi}{d\xi_r} \cdot \frac{a_r'^2}{a_r^3 \sqrt{(\Sigma a_r'^2)}} + \Sigma \frac{d^2 \phi}{d\xi_r^2} \cdot \frac{h^2}{a_r^2 \cdot \Sigma a_r'^2} + 2(\Sigma\Sigma) \frac{d^2 \phi}{d\xi_r d\xi_{r'}} \cdot \frac{h^2}{a_r a_{r'} \Sigma a_r'^2},$$

( $\Sigma\Sigma$ ) as before comprising all the  $\frac{s \cdot s - 1}{1 \cdot 2}$  combinations of the  $s$  indices taken in pairs.

Hence, the quantity on the right side of the equation (17), when we make  $H = H_0$ , becomes

$$\begin{aligned} & \Sigma \frac{d\phi}{d\xi_r} \cdot \frac{a_r'^2}{a_r^3 \sqrt{(\Sigma a_r'^2)}} + \Sigma \frac{d^2\phi}{d\xi_r^2} \cdot \frac{h^2}{a_r^2 \Sigma a_r'^2} + \kappa \phi \\ & + 2(\Sigma\Sigma) \frac{d^2\phi}{d\xi_r d\xi_{r'}} \cdot \frac{h^2}{a_r a_{r'} \Sigma a_r'^2} + \left( n - \Sigma \frac{a_r'^2}{a_r^2} \right) \Sigma \frac{d\phi}{d\xi_r} \cdot \frac{1}{a_r \sqrt{(\Sigma a_r'^2)}} \dots\dots (23). \end{aligned}$$

But if we recollect that we have generally

$$\xi_r = \frac{a_r}{\sqrt{(\Sigma a_r'^2)}} \dots\dots (24),$$

it is easy to perceive that in consequence of the equation (18) the quantity (23) will vanish, and therefore the foregoing value of  $H_0$  will always satisfy the equation (17).

Having thus a particular value of  $H$ , we immediately get the general one by assuming

$$H = H_0 \int \kappa dh.$$

In fact, there thence results

$$H = K H_0 \int \frac{h^{s-n} dh}{H_0^2 a_1, a_2, a_3, \dots, a_s};$$

the two arbitrary constants which the general integral ought to contain being  $K$ , and that which enters implicitly into the indefinite integral. But the condition  $0 = V''$  requires that  $H$  should vanish when  $h$  is infinite, and consequently the particular value adapted to the present investigation is

$$H_0 = K \cdot H_0 \int_{\infty} \frac{h^{s-n} dh}{H_0^2 a_1, a_2, \dots, a_s}.$$

11. The values of  $\phi$  and  $H$  being known, we may readily find the corresponding values of  $V$  and  $\rho'$ . For we have immediately

$$V = H\phi = K\phi H_0 \int_{\infty} \frac{h^{n-1} dh}{H_0^2 a_1, a_2, \dots, a_s} \dots \dots (26),$$

and as the function  $\phi$  is rational and entire, and the partial differential of  $V$  relative to  $h$  is finite, it follows that all the partial differentials of  $V$  are finite; and consequently, by what precedes (No. 7.) the condition (9') is satisfied by the foregoing value of  $V$ , as well as the equation (2) and condition  $0 = V''$ . Hence the equations (b) and (c) No. 6 will give, since

$$\frac{dV}{du} = -v \left( 1 - \sum_1^{s+1} \frac{a_r'^2 \xi_r^2}{a_r^2} \right)^{-1} \cdot \left\{ \sum_1^{s+1} \frac{h \xi_r}{a_r^2} \cdot \frac{dV}{d\xi_r} - \frac{dV}{dh} \right\},$$

and  $h$  must be supposed equal to zero in these equations

$$\rho' = \frac{-\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{s}{2}} \Gamma\left(\frac{n-s+1}{2}\right)} v^{n-s-1} \cdot h^{n-s} \frac{dV}{dh} \dots \dots (\text{where } h = 0):$$

since where  $h = 0$ ,  $a_r = a_r'$ ; and therefore,

$$1 - \sum_1^{s+1} \frac{a_r'^2 \xi_r^2}{a_r^2} = 1 - \sum_1^{s+1} \xi_r^2 = v^2.$$

If now we substitute for  $V$  its value (26), and recollect that  $n-s+1$  is always positive, we get

$$\rho' = \frac{-\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{s}{2}} \Gamma\left(\frac{n-s+1}{2}\right)} v^{n-s-1} \phi' \frac{K}{H_0' a_1', a_2', \dots, a_s'} \dots \dots (27),$$

since it is clear from the form of  $H_0$  that this quantity may always be expanded in a series of the entire powers of  $h^2$ . In the preceding expression, (27),  $H_0'$  indicates the value of  $H_0$  when  $h = 0$ , and  $\phi'$  the corresponding value of  $\phi$  or that which would be obtained by simply changing the unaccented letter  $\xi_1, \xi_2, \dots, \xi_s$  into the accented ones  $\xi_1', \xi_2', \dots, \xi_s'$  deduced from

$$(\gamma) \quad x_1' = a_1' \xi_1'; \quad x_2' = a_2' \xi_2'; \quad x_s' = a_s' \xi_s'.$$

It will now be easy to obtain the value of  $V$  corresponding to

$$\rho' = \left(1 - \frac{x_1'^2}{a_1'^2} - \frac{x_2'^2}{a_2'^2} - \dots - \frac{x_s'^2}{a_s'^2}\right)^{\frac{n-s-1}{2}} F(x_1', x_2', \dots, x_s') \dots (28),$$

without integrating the formula (1) No 1, where  $F$  is the characteristic of any rational and entire function. In fact it is easy to see from what precedes (No. 9), that we may always expand  $F$  in a finite series of the form

$$F(x_1', x_2', \dots, x_s') = b_0 \phi_0' + b_1 \phi_1' + b_2 \phi_2' + b_3 \phi_3' + \&c.$$

after  $x_1', x_2', \&c.$  have been replaced with their values  $(\gamma)$ . Hence, we immediately get

$$\rho' = v^{n-s-1} \cdot \{b_0 \phi_0' + b_1 \phi_1' + b_2 \phi_2' + \&c.\} \dots (29).$$

By comparing the formulæ (26) and (27) it is clear that any term, as  $b_r \phi_r'$  for instance, of the series entering into  $\rho'$ , will have for corresponding term in the required value of  $V$ , the quantity

$$- \frac{2\pi^{\frac{s}{2}} \Gamma\left(\frac{n-s+1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} H_0' a_1' a_2' \dots a_s' \cdot b_r \phi_r H_0 \int_{\infty}^{\infty} \frac{h^{s-n} dh}{H_0^2 a_1 a_2 \dots a_s} \dots (30):$$

$H_0$  being a particular value of  $H$  satisfying the equation (17), and immediately deducible from  $\phi$  by the method before explained.

12. All that now remains, is to demonstrate that

$$\nabla' \nabla \phi = \nabla \nabla' \phi \dots (31),$$

whatever  $\phi$  may be. For this purpose let us here resume the value of  $\Delta \phi$ , as immediately deduced from the equation (2'') No. 7, viz.

$$\begin{aligned} \Delta \phi &= (1 - \Sigma \xi^2) \Sigma \frac{d^2 \phi}{a^2 d\xi^2} + (s-n-1) \Sigma \frac{\xi d\phi}{a^2 d\xi} \\ &+ h^2 \Sigma \frac{\xi^2}{a^2} \times \Sigma \frac{d^2 \phi}{a^2 d\xi^2} - h^2 \Sigma \Sigma \frac{\xi \xi'}{a^2 a'^2} \frac{d^2 \phi}{d\xi d\xi'} \\ &+ h^2 \Sigma \frac{\xi d\phi}{a^4 d\xi} - h^2 \Sigma \frac{1}{a_2} \times \Sigma \frac{\xi d\phi}{a^2 d\xi} \dots (32), \end{aligned}$$

where for simplicity the indices at the foot of the letters  $\xi$  and  $a$  have been omitted, and their accents transferred to the letters themselves. Moreover all the finite integrals are supposed taken from 1 to  $s+1$ .

By making  $h = 0$  in the last expression we immediately get  $\nabla' \phi$ , and if for a moment, to prevent ambiguity, we write  $b_r$  in the place of the original  $a'_r$  and omit the lower indices as before, we obtain

$$\nabla' \phi = (1 - \Sigma \xi^2) \Sigma \frac{d^2 \phi}{b''^2 d \xi''^2} + (s - n - 1) \Sigma \frac{\xi'' d \phi}{b''^2 d \xi''} \dots\dots (33);$$

where to avoid all risk of confusion  $r$  has been changed into  $r''$ , and the double accent of this index transferred to the letters  $\xi$  and  $b$  themselves.

We will now conceive the expression (32) to be written in the abridged form

$$\nabla \phi = \nabla_1 \phi + h^2 \nabla_2 \phi - h^2 \nabla_3 \phi + h^2 \nabla_4 \phi - h^2 \nabla_5 \phi,$$

the order of the terms remaining unchanged.

If then we recollect that the accents have no other office to perform than to keep the various finite integrations quite distinct, and consequently that in the final results they may be permuted in any way at will, we shall readily get

$$\begin{aligned} \nabla' \nabla_1 \phi - \nabla_1 \nabla' \phi = \\ (1 - \Sigma \xi^2) \left\{ 4 \Sigma \Sigma \left( \frac{1}{a'^2 b^2} - \frac{1}{a^2 b'^2} \right) \frac{\xi' d^3 \phi}{d \xi^2 d \xi'} \right. & (1) + 2 \Sigma \frac{1}{a^2} \times \Sigma \frac{d^3 \phi}{b^2 d \xi^2} & (2) - 2 \Sigma \frac{1}{b^2} \times \Sigma \frac{d^3 \phi}{a^2 d \xi^2} & (3) \left. \right\} \\ + 2 (s - n - 1) \left\{ \Sigma \frac{\xi^2}{a^2} \times \Sigma \frac{d^2 \phi}{b^2 d \xi^2} \right. & (4) - \Sigma \frac{\xi^2}{b^2} \times \Sigma \frac{d^2 \phi}{a^2 d \xi^2} & (5) \left. \right\}, \\ \nabla' \nabla_2 \phi - \nabla_2 \nabla' \phi = \\ (1 - \Sigma \xi^2) \left\{ 4 \Sigma \Sigma \frac{\xi'}{a^2 a'^2 b'^2} \cdot \frac{d^3 \phi}{d \xi^2 d \xi'} \right. & (6) + 2 \Sigma \frac{1}{a^2 b^2} \times \Sigma \frac{d^2 \phi}{a^2 d \xi^2} & (7) \left. \right\} \\ + 4 \Sigma \frac{\xi^2}{a^2} \times \Sigma \Sigma \frac{\xi}{a^2 b'^2} \cdot \frac{d^3 \phi}{d \xi d \xi'^2} & (8) + 2 \Sigma \frac{1}{a^2} \times \Sigma \frac{\xi^2}{a^2} \times \Sigma \frac{d^2 \phi}{b^2 d \xi^2} & (9) \end{aligned}$$

$$\begin{aligned}
& + 2(s-n-1) \left\{ \Sigma \frac{\xi^2}{a^2 b^2} \times \Sigma \frac{d^2 \phi}{a^2 d\xi^2} \right. \quad (10) \quad \left. - \Sigma \frac{\xi^2}{a^2} \times \Sigma \frac{1}{a^2 b^2} \cdot \frac{d^2 \phi}{d\xi^2} \right. \quad (11) \quad \left. \right\} \\
& \quad \nabla_3 \nabla' \phi - \nabla' \nabla_3 \phi = \\
& \quad (1 - \Sigma \xi^2) \left\{ -4 \Sigma \Sigma \frac{\xi d^3 \phi}{a^2 a'^2 b'^2 d\xi d\xi'^2} \right. \quad (12) \quad \left. - 2 \Sigma \frac{1}{a^4 b^2} \cdot \frac{d^2 \phi}{d\xi^2} \right. \quad (13) \quad \left. \right\} \\
& \quad - 4 \Sigma \frac{\xi^2}{a^2} \times \Sigma \Sigma \frac{\xi}{a^2 b'^2} \cdot \frac{d^3 \phi}{d\xi d\xi'^2} \quad (14) \quad - 2 \Sigma \frac{\xi^2}{a^4} \times \Sigma \frac{d^2 \phi}{b^2 d\xi^2} \quad (15) \\
& \quad \nabla' \nabla_4 \phi - \nabla_4 \nabla' \phi = \\
& \quad 2(1 - \Sigma \xi^2) \Sigma \frac{1}{b^2 a^4} \cdot \frac{d^2 \phi}{d\xi^2} \quad (16) \quad + 2 \Sigma \frac{\xi^2}{a^4} \times \Sigma \frac{d^2 \phi}{b^2 d\xi^2} \quad (17) \\
& \quad \nabla_5 \nabla' \phi - \nabla' \nabla_5 \phi = \\
& \quad - 2 \cdot (1 - \Sigma \xi^2) \Sigma \frac{1}{a^2} \times \Sigma \frac{1}{a^2 b^2} \cdot \frac{d^2 \phi}{d\xi^2} \quad (18) \quad - 2 \cdot \Sigma \frac{1}{a^2} \times \Sigma \frac{\xi^2}{a^2} \times \Sigma \frac{d^2 \phi}{b^2 d\xi^2} \quad (19)
\end{aligned}$$

all the finite integrals being taken from  $r = 1$  to  $r = s + 1$ , and from  $r' = 1$  to  $r' = s + 1$ .

In order to obtain the required value

$$\nabla' \nabla \phi - \nabla \nabla' \phi,$$

it is clear that we shall only have to add the first of the five preceding quantities to the sum of the four following ones multiplied by  $h^2$ , and to render this more easy, we have appended to each of the terms in the preceding quantities a number inclosed in a small parenthesis.

Now since the accents may be permuted at will, and we have likewise  $a^2 = b^2 + h^2$ , it is easy to see that the terms marked (1), (6) and (12) mutually destroy each other. In like manner, (2), (3), (7) and (18) mutually destroy each other; the same may evidently be said of (13) and (16), of (15) and (17), of (9) and (19), and of (8) and (14). Moreover the four quantities (4), (5), (10) and (11) will do so likewise, and consequently, we have

$$\nabla' \nabla \phi - \nabla \nabla' \phi = 0.$$

Hence the truth of the equation (31) is manifest.

*Application of the preceding General Theory to the Determination of the Attractions of Ellipsoids.*

13. Suppose it is required to determine the attractions exerted by an ellipsoid whose semi-axes are  $a', b', c'$  whether the attracted point  $p$  is situated within the ellipsoid or not, the law of the attraction being inversely as the  $n^{\text{th}}$  power of the distance. Then it is well known that the required attractions may always be deduced from the function

$$V = \int \frac{\rho' dx' dy' dz'}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{n'-1}{2}}};$$

$\rho'$  being the density of the element  $dx' dy' dz'$  of the ellipsoid, and  $x, y, z$  being the rectangular co-ordinates of  $p$ .

We may avoid the breach of the law of continuity which takes place in the value of  $V$ , when the point  $p$  passes from the interior of the ellipsoid into the exterior space, by adding the positive quantity  $u^2$  to that inclosed in the braces, and may afterwards suppose  $u$  evanescent in the final result. Let us therefore now consider the function,

$$V = \int \frac{\rho' dx' dy' dz'}{\{(x-x')^2 + (y-y')^2 + (z-z')^2 + u^2\}^{\frac{n'-1}{2}}};$$

this triple integral like the preceding including all the values of  $x', y', z'$ , admitted by the condition

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} + \frac{z'^2}{c'^2} < 1.$$

If now we suppose the density  $\rho'$  is of the form

$$\rho' = \left(1 - \frac{x'^2}{a'^2} - \frac{y'^2}{b'^2} - \frac{z'^2}{c'^2}\right)^{\frac{n'-2}{2}} f(x', y', z') \dots \dots \dots (34),$$

which will simplify  $f(x', y', z')$  when  $\rho'$  is constant and  $n' = 2$ , and then compare this value with the one immediately deducible from the general expression (28) by supposing for a moment  $n' = n$ , viz.

$$\rho' = \left(1 - \frac{x'^2}{a'^2} - \frac{y'^2}{b'^2} - \frac{z'^2}{c'^2}\right)^{\frac{n'-4}{2}} F(x', y', z'),$$

we see that the function  $f$  will always be two degrees higher than  $F$ . But since our formulæ become more complicated in proportion as the degree of  $F$  is higher, it will be simpler to determine the differentials of  $V$ , because for these differentials the degree of  $F$  and  $f$  is the same. Let us therefore make

$$A = \frac{1}{1-n'} \frac{dV}{dx} = \int \frac{\rho' (x-x') dx' dy' dz'}{\{(x-x')^2 + (y-y')^2 + (z-z')^2 + u^2\}^{\frac{n'+1}{2}}},$$

then this quantity naturally divides itself into two parts, such that

$$A = xA' + A'',$$

$$\text{where } A' = + \int \frac{\rho' dx' dy' dz'}{\{(x-x')^2 + (y-y')^2 + (z-z')^2 + u^2\}^{\frac{n'+1}{2}}},$$

$$\text{and } A'' = - \int \frac{x' \rho' dx' dy' dz'}{\{(x-x')^2 + (y-y')^2 + (z-z')^2 + u^2\}^{\frac{n'+1}{2}}}.$$

By comparing these with the general formula (1), it is clear that  $n-1 = n'+1$ , and consequently  $n = n'+2$ . In this way the expression (28) gives

$$\rho' = \left(1 - \frac{x'^2}{a'^2} - \frac{y'^2}{b'^2} - \frac{z'^2}{c'^2}\right)^{\frac{n'-2}{2}} F'(x', y', z'),$$

which coincides with (34) by supposing  $F=f$ .

The simplest case of the present theory is where  $f(x', y', z')=1$ , and then by No 11, we have  $\phi'_0=1$  and  $b_0=1$ , when  $A'$  is the quantity required, and as the general series (29), No 11, then reduces itself to its first term, we immediately obtain from the formula (30), the value of  $A'$  following,

$$A' = - \frac{2\pi^{\frac{3}{2}} \Gamma\left(\frac{n'}{2}\right)}{\Gamma\left(\frac{n'+1}{2}\right)} a'b'c' \int_{\infty}^{\frac{h^{1-n'}}{abc}} \frac{h^{1-n'} dh}{abc} \dots\dots\dots (35),$$

because in the present case  $H_0=1$ ,  $s=3$ , and  $n=n'+2$ .

Again, the same general theory being applied to the value of  $A''$  given above, we get

$$F(x', y', z') = -x' f(x', y', z') = -x' \quad (\text{when } f=1),$$

and hence by No 11,  $F(x', y', z') = -a' \xi'$ . In this way the series (29) again reduces itself to a single term, in which

$$\phi'_0 = \xi', \quad \text{and } b_0 = -a',$$

and the particular value  $H_0$  corresponding thereto, by omitting the superfluous constant  $\frac{1}{\sqrt{(a'^2 + b'^2 + c'^2)}}$  will be (No 10),

$$H_0 = a.$$

These substituted in the general formula (30) as before, immediately give

$$A'' = + \frac{2\pi^{\frac{3}{2}} \Gamma\left(\frac{n'}{2}\right)}{\Gamma\left(\frac{n'+1}{2}\right)} a'^3 b' c' \xi a \int_{\infty}^{\infty} \frac{h^{1-n'} dh}{a^3 b c},$$

and consequently by reduction since  $a \xi = x$ ,

$$A = x A' + A'' = - \frac{2\pi^{\frac{3}{2}} \Gamma\left(\frac{n'}{2}\right)}{\Gamma\left(\frac{n'+1}{2}\right)} a' b' c' x \int_{\infty}^{\infty} \frac{h^{3-n'} dh}{a^3 b c} \dots\dots\dots(36).$$

The value of  $A$  just given belongs to the density

$$\rho' = \left(1 - \frac{x'^2}{a'^2} - \frac{y'^2}{b'^2} - \frac{z'^2}{c'^2}\right)^{\frac{n'-2}{2}}.$$

Hence we immediately obtain without calculation the corresponding values

$$B = \frac{1}{1-n'} \frac{dV}{dy} = - \frac{2\pi^{\frac{3}{2}} \Gamma\left(\frac{n'}{2}\right)}{\Gamma\left(\frac{n'+1}{2}\right)} a' b' c' y \int_{\infty}^{\infty} \frac{h^{3-n'} dh}{a b^3 c},$$

$$C = \frac{1}{1-n'} \frac{dV}{dz} = - \frac{2\pi^{\frac{3}{2}} \Gamma\left(\frac{n'}{2}\right)}{\Gamma\left(\frac{n'+1}{2}\right)} a' b' c' z \int_{\infty}^{\infty} \frac{h^{3-n'} dh}{a b c^3}.$$

If now we suppose moreover

$$D = \frac{1}{1-n'} \frac{dV}{du} = u \int \frac{\rho' dx' dy' dz'}{\{(x-x')^2 + (y-y')^2 + (z-z')^2 + u^2\}^{\frac{n'+1}{2}}},$$

the method before explained (No 11), will immediately give

$$D = - \frac{2\pi^{\frac{3}{2}} \Gamma\left(\frac{n'}{2}\right)}{\Gamma\left(\frac{n'+1}{2}\right)} a' b' c' u \int_{\infty}^h \frac{h^{1-n'} dh}{abc},$$

and therefore if for abridgment we make

$$M = (n'-1) \frac{\pi^{\frac{3}{2}} \Gamma\left(\frac{n'}{2}\right)}{\Gamma\left(\frac{n'+1}{2}\right)} a' b' c',$$

the total differential of  $V$  may be written

$$dV = M \left\{ 2x dx \int_{\infty}^h \frac{h^{3-n'} dh}{a^3 b c} + 2y dy \int_{\infty}^h \frac{h^{3-n'} dh}{a b^3 c} + 2z dz \int_{\infty}^h \frac{h^{3-n'} dh}{a b c^3} + 2u du \int_{\infty}^h \frac{h^{1-n'} dh}{a b c} \right\},$$

which being integrated in the usual way by first supposing  $h$  constant, and then completing the integral with a function of  $h$ , to be afterwards determined by making every thing in  $V$  variable, we get

$$V = M \left\{ x^2 \int_{\infty}^h \frac{h^{3-n'} dh}{a^3 b c} + y^2 \int_{\infty}^h \frac{h^{3-n'} dh}{a b^3 c} + z^2 \int_{\infty}^h \frac{h^{3-n'} dh}{a b c^3} + u^2 \int_{\infty}^h \frac{h^{1-n'} dh}{a b c} \right\} + k;$$

$k$  being a quantity absolutely constant, which is equal to zero when  $n' > 1$ . What has just been advanced will be quite clear if we recollect that  $h$  may be regarded as a function of  $x, y, z$  and  $u$ , determined by the equation

$$1 = \frac{x^2}{a'^2 + h^2} + \frac{y^2}{b'^2 + h^2} + \frac{z^2}{c'^2 + h^2} + \frac{u^2}{h^2} = \zeta^2 + \eta^2 + \zeta'^2 + v^2 \dots \dots \dots (37);$$

seeing that  $a^2 = a'^2 + h^2$ ,  $b^2 = b'^2 + h^2$ , and  $c^2 = c'^2 + h^2$ .

After what precedes, it seems needless to enter into an examination of the values of  $V$  belonging to other values of the density  $\rho'$ , since it must be clear that the general method is equally applicable when

$$\rho' = \left(1 - \frac{x'^2}{a'^2} - \frac{y'^2}{b'^2} - \frac{z'^2}{c'^2}\right)^{\frac{n'-2}{2}} f(x', y', z');$$

where  $f$  is the characteristic of any rational and entire function.

The quantity  $A$  before determined when we make  $u = 0$ , serves to express the attraction in the direction of the co-ordinate  $x$  of an ellipsoid on any point  $p$ , situated at will either within or without it. But by making  $u = 0$  in (37) we have

$$1 = \frac{x^2}{a'^2 + h^2} + \frac{y^2}{b'^2 + h^2} + \frac{z^2}{c'^2 + h^2} + \frac{o^2}{h^2} \dots \dots \dots (38),$$

and it is thence easy to perceive that when  $p$  is within the ellipsoid,  $h$  must constantly remain equal to zero, and the equation (38) will always be satisfied by the indeterminate positive quantity  $\frac{o^2}{o^2}$ . When on the contrary  $p$  is exterior to it,  $h$  can no longer remain equal to zero, but must be such a function of  $x, y, z$ , as will satisfy the equation (38), of which the last term now evidently vanishes in consequence of the numerator  $o^2$ . Thus the forms of the quantities  $A, B, C, D$  and  $V$  all remain unchanged, and the discontinuity in each of them falls upon the quantity  $h$ .

To compare the value of  $A$  here found with that obtained by the ordinary methods, we shall simply have to make  $n' = 2$  in the expression (36), recollecting that  $\Gamma(1) = 1$ , and  $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$ . In this way

$$\begin{aligned} A &= -4\pi a'b'c'x \int_{\infty}^{\frac{h}{a^3bc}} \frac{h dh}{a^3bc} = -4\pi a'b'c'x \int_{\infty}^{\frac{da}{a^2bc}} \frac{da}{a^2bc} \\ &= +4a'b'c'x \int_a^{\infty} \frac{da}{a^2bc} = 4\pi a'b'c' \int_a^{\infty} \frac{da}{a^2 \sqrt{(a^2 - a'^2 + b'^2)(a^2 - a'^2 + c'^2)}}. \end{aligned}$$

But the last quantity may easily be put under the form of a definite integral, by writing  $\frac{a}{v}$  in the place of  $a$  under the sign of integration, and again inverting the limits. Thus there will result

$$A = \frac{4\pi a'b'c'}{a^3} \int_0^1 \frac{v^2 dv}{\sqrt{\left(1 + \frac{b'^2 - a'^2}{a^2} v^2\right) \left(1 + \frac{c'^2 - a'^2}{a^2} v^2\right)}},$$

which agrees with the ordinary formula, since the mass of the ellipsoid is  $\frac{4\pi a'b'c'}{3}$  and  $a^2 = a'^2 + h^2$ .

*Examination of a particular Case of the General Theory exposed in the former Part of this Paper.*

14. There is a particular case of the general theory first considered, which merits notice, in consequence of the simplicity of the results to which it leads. The case in question is that where we have generally whatever  $r$  may be

$$a'_r = a'.$$

Then the equation (19) which serves to determine  $\phi$ , becomes by supposing  $k_0 = k.a'^2$

$$0 = (1 - \Sigma_1^{s+1} \xi_r^2) \Sigma_1^{s+1} \frac{d^2 \phi}{d\xi_r^2} + (s-n-1) \Sigma_1^{s+1} \xi_r \frac{d\phi}{d\xi_r} - k\phi \dots\dots\dots(39).$$

If now we employ a transformation similar to that used in obtaining the formula (14), No 6, by making

$$\xi_1 = \rho \cos \theta_1, \xi_2 = \rho \sin \theta_1 \cos \theta_2, \xi_3 = \rho \sin \theta_1 \sin \theta_2 \cos \theta_3, \&c.$$

and then conceive the equation (39) deduced from the condition that

$$\int d\xi_1 d\xi_2 \dots\dots d\xi_s (1 - \Sigma \xi_r^2)^{\frac{n-s+1}{2}} \left\{ \Sigma_1^{s+1} \left( \frac{d\phi}{d\xi_r} \right)^2 + \frac{k\phi^2}{1 - \Sigma \xi_r^2} \right\}$$

must be a minimum (vide No 8), we shall have

$$d\xi_1 d\xi_2 \dots\dots d\xi_s = \rho^{s-1} \sin \theta_1^{s-2} \sin \theta_2^{s-3} \dots\dots \sin \theta_{s-2} d\rho d\theta_1 d\theta_2 \dots\dots d\theta_{s-1},$$

$$\Sigma_1^{s+1} \left( \frac{d\phi}{d\xi_r} \right)^2 = \left( \frac{d\phi}{d\rho} \right)^2 + \frac{1}{\rho^2} \Sigma_1^s \frac{\left( \frac{d\phi}{d\xi_r} \right)^2}{\sin \theta_1^2 \sin \theta_2^2 \dots\dots \sin \theta_{r-1}^2},$$

$$\text{and } 1 - \Sigma \xi_r^2 = 1 - \rho^2.$$

Proceeding now in the manner before explained, (No 8), we obtain for the equivalent of (39) by reduction

$$0 = \frac{d^2 \phi}{d\rho^2} + \frac{s-1-n\rho^2}{\rho(1-\rho^2)} \cdot \frac{d\phi}{d\rho} + \frac{1}{\rho^2} \Sigma_1^s \frac{\frac{d^2 \phi}{d\theta_r^2} + (s-r-1) \frac{\cos \theta_r}{\sin \theta_r} \frac{d\phi}{d\theta_r}}{\sin \theta_1^2 \sin \theta_2^2 \dots\dots \sin \theta_{r-1}^2} - \frac{k}{1-\rho^2} \phi \dots\dots(40).$$

But this equation may be satisfied by a function of the form

$$\phi = P\Theta_1\Theta_2\Theta_3\dots\dots\dots\Theta_{s-1};$$

$P$  being a function of  $\rho$  only, and afterwards generally  $\Theta_r$  a function of  $\theta_r$  only. In fact, if we substitute this value of  $\phi$  in (40), and then divide the result by  $\phi$ , it is clear that it will be satisfied by the system

$$\begin{aligned} \frac{d^2\Theta_{s-1}}{\Theta_{s-1}d\theta_{s-1}^2} &= \lambda_{s-1} \\ \frac{d^2\Theta_{s-2}}{\Theta_{s-2}d\theta_{s-2}^2} + 1 \cdot \frac{\cos\theta_{s-2}}{\sin\theta_{s-2}} \frac{d\Theta_{s-2}}{\Theta_{s-2}d\theta_{s-2}} + \frac{\lambda_{s-1}}{\sin^2\theta_{s-2}} &= \lambda_{s-2} \dots\dots\dots(41), \\ \frac{d^2\Theta_{s-3}}{\Theta_{s-3}d\theta_{s-3}^2} + 2 \cdot \frac{\cos\theta_{s-3}}{\sin\theta_{s-3}} \frac{d\Theta_{s-3}}{\Theta_{s-3}d\theta_{s-3}} + \frac{\lambda_{s-2}}{\sin^2\theta_{s-3}} &= \lambda_{s-3} \\ \&c. \qquad \qquad \&c. \qquad \qquad \&c. \qquad \qquad \&c. \end{aligned}$$

combined with the following equation,

$$\frac{d^2P}{Pd\rho^2} + \frac{s-1-n\rho^2}{\rho(1-\rho^2)} \cdot \frac{dP}{Pd\rho} + \frac{\lambda_1}{\rho^2} - \frac{k}{1-\rho^2} = 0 \dots\dots\dots(42),$$

where  $k, \lambda_1, \lambda_2, \lambda_3, \&c.$  are constant quantities.

In order to resolve the system (41), let us here consider the general type of the equations therein contained, viz.

$$0 = \frac{d^2\Theta_{s-r}}{d\theta_{s-r}^2} + (r-1) \frac{\cos\theta_{s-r}}{\sin\theta_{s-r}} \cdot \frac{d\Theta_{s-r}}{d\theta_{s-r}} + \left( \frac{\lambda_{s-r+1}}{\sin^2\theta_{s-r}} - \lambda_{s-r} \right) \Theta_{s-r}.$$

Now if we reflect on the nature of the results obtained in a preceding part of this paper, it will not be difficult to see that  $\Theta_{s-r}$  is of the form

$$\Theta_{s-r} = (\sin\theta_{s-r})^i p = (1-\mu^2)^{\frac{i}{2}} p;$$

where  $p$  is a rational and entire function of  $\mu = \cos\theta_{s-r}$ , and  $i$  a whole number.

By substituting this value in the general type and making

$$\lambda_{s-r+1} = -i(i+r-2) \dots\dots\dots(43)$$

we readily obtain

$$0 = (1-\mu^2) \frac{d^2p}{d\mu^2} - (2i+r)\mu \frac{dp}{d\mu} - \{\lambda_{s-r} + i(i+r-1)\} p.$$

To satisfy this equation, let us assume

$$p = \sum_0^\infty A_t \mu^{e-i-2t}.$$

Then by substituting in the above and equating separately the coefficients of the various powers of  $\mu$ , we have in the first place from the highest

$$\lambda_{s-r} = -e(e+r-1) \dots\dots\dots (44),$$

and afterwards generally

$$A_{t+1} = -\frac{e-i-2t \cdot e-i-2t-1}{2t+2 \times 2e+r-2t-3} A_t.$$

But the equation (43) may evidently be made to coincide with (44), by writing  $i^{(r)}$  for  $i$ , and  $i^{(r+1)}$  for  $e$ , since then both will be comprised in

$$\lambda_{s-r+1} = -i^{(r)} \{i^{(r)} + r - 2\} \dots\dots\dots (45).$$

Hence we readily get for the general solution of the system (41),

$$\begin{aligned} \Theta_{s-r} = & (1 - \mu^2)^{\frac{i^{(r)}}{2}} \left[ \mu^{i^{(r+1)} - i^{(r)}} - \frac{\{i^{(r+1)} - i^{(r)}\} \{i^{(r+1)} - i^{(r)} - 1\}}{2 \times 2i^{(r)} + r - 3} \mu^{i^{(r+1)} - i^{(r)} - 2} \right. \\ & + \frac{\{i^{(r+1)} - i^{(r)}\} \{i^{(r+1)} - i^{(r)} - 1\} \{i^{(r+1)} - i^{(r)} - 2\} \{i^{(r+1)} - i^{(r)} - 3\}}{2 \cdot 4 \times \{2i^{(r)} + r - 3\} \{2i^{(r)} + r - 5\}} \mu^{i^{(r+1)} - i^{(r)} - 4} - \&c. \left. \right]; \end{aligned}$$

where  $\mu = \cos \theta_{s-r}$ , and  $i^{(r)}$  represents any positive integer whatever, provided  $i^{(r)}$  is never greater than  $i^{(r+1)}$ .

Though we have thus the solution of every equation in the system (41), yet that of the first may be obtained under a simpler form by writing therein for  $\lambda_{s-1}$  its value  $-i^{(2)^2}$  deduced from (45). We shall then immediately perceive that it is satisfied by

$$\Theta_{s-1} = \frac{\sin}{\cos} \left\{ i^{(2)} \theta_{s-1} \right\}.$$

In consequence of the formula (45), the equation (42) becomes

$$0 = \frac{d^2 P}{d\rho^2} + \frac{s-1-n\rho^2}{\rho(1-\rho^2)} \cdot \frac{dP}{d\rho} - \left\{ \frac{i^{(s)}(i^{(s)}+s-2)}{\rho^2} + \frac{k}{1-\rho^2} \right\} P,$$

which is satisfied by making  $k = -\lambda_1 - (i^{(s)} + 2\omega)(i^{(s)} + 2\omega + n - 1)$ , and

$$P = \rho^{i(s)} \left\{ \rho^{2\omega} - \frac{2\omega \times 2i^{(s)} + 2\omega + s - 2}{2 \times 2i^{(s)} + 4\omega + n - 3} \rho^{2\omega-2} \right. \\ \left. + \frac{2\omega \cdot 2\omega - 2 \times 2i^{(s)} + 2\omega + s - 2 \cdot 2i^{(s)} + 2\omega + s - 4}{2 \cdot 4 \times 2i^{(s)} + 4\omega + n - 3 \cdot 2i + 4\omega + n - 5} \rho^{2\omega-4} - \&c. \right\}$$

where  $\omega$  represents any whole positive number.

Having thus determined all the factors of  $\phi$ , it now only remains to deduce the corresponding value of  $H$ . But  $H_0$  the particular value satisfying the differential equation in  $H$ , will be had from  $\phi$  by simply making therein

$$\xi_r = \frac{a_r}{\sqrt{(\sum a_r'^2)}} = \frac{a}{a' \sqrt{s}},$$

since in the present case we have generally  $a_r' = a'$ .

Hence, it is clear that the proper values of  $\theta_1, \theta_2, \theta_3$ , &c. to be here employed are all constant, and consequently the factor

$$\Theta_1 \Theta_2 \Theta_3 \dots \Theta_{s-1}$$

entering into  $\phi$  is likewise constant. Neglecting therefore this factor as superfluous, we get for the particular value of  $H$ ,

$$H_0 = P_{\frac{a}{a'}};$$

$$\text{since } \rho^2 = \xi_1^2 + \xi_2^2 + \dots + \xi_s^2 = \frac{s a^2}{s a'^2} = \frac{a^2}{a'^2},$$

and  $P_{\frac{a}{a'}}$  represents what  $P$  becomes when  $\rho$  is changed into  $\frac{a}{a'}$ .

Substituting this value of  $H_0$  in the equation (25), No 10, there results since  $a^2 = a'^2 + h^2$

$$H = K \cdot P_{\frac{a}{a'}} \int_{\infty}^{\frac{h^{s-n} dh}{P_{\frac{a}{a'}}^2 (a'^2 + h^2)^{\frac{s}{2}}}} \dots \dots \dots (46)$$

$K$  being an arbitrary constant quantity.

Thus the complete value of  $V$  for the particular case considered in the present number is

$$V = P\Theta_1\Theta_2\dots\Theta_{s-1} \cdot K P_{\frac{a}{a}} \int_{\infty} \frac{h^{s-n} dh}{P_{\frac{a}{a}}^2 (a'^2 + h^2)^{\frac{s}{2}}} \dots\dots\dots (47)$$

and the equation (27), No 11, will give for the corresponding value of  $\rho'$ ,

$$\rho' = \frac{-\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{s}{2}}\Gamma\left(\frac{n-s+1}{2}\right)} (1-\rho^2)^{\frac{n-s-1}{2}} \frac{K}{P_1 a'^s} P'\Theta_1'\Theta_2'\dots\Theta_{s-1}'\dots\dots\dots (48);$$

where  $P_1', \Theta_1', \Theta_2', \&c.$  are the values which the functions  $P, \Theta_1, \Theta_2, \&c.$  take when we change the unaccented variables  $\xi_1, \xi_2, \dots, \xi_s$  into the corresponding accented ones  $\xi_1', \xi_2', \dots, \xi_s'$ , and

$$P_1 = \frac{n-s+1 \cdot n-s+3 \cdot \dots \cdot n-s+2\omega-1}{n+2i+2\omega-1 \cdot n+2i+2\omega+1 \cdot \dots \cdot n+2i+4\omega-3},$$

or the value of  $P$  when  $\rho = 1$ ; where as well as in what follows  $i$  is written in the place of  $i^{(s)}$ .

The differential equation which serves to determine  $H$  when we introduce  $a$  instead of  $h$  as independent variable, may in the present case be written under the form

$$0 = a^2(a^2 - a'^2) \frac{d^2 H}{da^2} + a^2 \{na^2 - (s-1) \cdot a'^2\} \frac{dH}{ada} \\ + \{i(i+s-2)a'^2 - (i+2\omega)(i+2\omega+n-1)a^2\} H,$$

and the particular integral here required is that which vanishes when  $h$  is infinite. Moreover it is easy to prove, by expanding in series, that this particular integral is

$$H = k' a^i \Delta^\omega \cdot a^{2r} \int_{\infty} a^{1-2r-s-2i} da (a^2 - a'^2)^{\frac{s-1-n-2\omega}{2}};$$

provided we make the variable  $r$  to which  $\Delta^\omega$  refers, vanish after all the operations have been effected.

But the constant  $k'$  may be determined by comparing the coefficient of the highest power of  $a$  in the expansion of the last formula with the like coefficient in that of the expression (46), and thus we have

$$k' = K a^{i+2\omega} (-1)^\omega \frac{n+2i+2\omega-1 \cdot n+2i+2\omega+1 \cdot \dots \cdot n+2i+4\omega-3}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2\omega}.$$

Hence we readily get for the equivalent of (47),

$$V = P \Theta_1 \Theta_2 \dots \Theta_{s-1} \times \frac{n+2i+2\omega-1 \cdot n+2i+2\omega+1 \cdot \dots \cdot n+2i+4\omega-3}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2\omega} \\ \dots \times K a'^{i+2\omega} (-1)^\omega \Delta^\omega a'^{2r} \int_{\infty}^a da a^{1-2r-s-2i} (a^2 - a'^2)^{\frac{s-1-n-2\omega}{2}}.$$

In certain cases the value of  $V$  just obtained will be found more convenient than the foregoing one (47). Suppose for instance we represent the value of  $V$  when  $h=0$ , or  $a=a'$  by  $V_0$ . Then we shall hence get

$$V_0 = P \Theta_1 \Theta_2 \dots \Theta_{s-1} \times \frac{n+2i+2\omega-1 \cdot n+2i+2\omega+1 \cdot \dots \cdot n+2i+4\omega-3}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2\omega} \\ \times K a'^{2i+2\omega} (-1)^\omega \cdot \Delta^\omega a'^{2r} \int_{\infty}^a da a^{1-2r-s-2i} (a^2 - a'^2)^{\frac{s-1-n-2\omega}{2}} \dots (\delta),$$

which in consequence of the well known formula

$$\int_{\infty}^a a^{-m} da (a^2 - a'^2)^{-p} = -a'^{1-m-2p} \times \frac{\Gamma(1-p) \Gamma\left(\frac{m+2p-1}{2}\right)}{2 \Gamma\left(\frac{1+m}{2}\right)},$$

by reduction becomes

$$V_0 = -P \Theta_1 \Theta_2 \dots \Theta_{s-1} \times \frac{\Gamma\left(\frac{1+s-n}{2}\right) \Gamma\left(\frac{n+2i+4\omega-1}{2}\right)}{2 \Gamma(\omega+1) \Gamma\left(\frac{s+2i+2\omega}{2}\right)} K a'^{1-n} \dots (49);$$

since in the formula ( $\delta$ ),  $r$  ought to be made equal to zero at the end of the process.

By conceiving the auxiliary variable  $u$  to vanish, it will become clear from what has been advanced in the preceding number, that the values of the function  $V$  within circular planes and spheres, are only particular cases of the more general one, (49), which answer to  $s=2$  and  $s=3$  respectively. We have thus by combining the expressions (48) and (49), the means of determining  $V_0$  when the density  $\rho'$  is given, and *vice versa*; and the present method of resolving these problems seems more simple if possible than that contained in the articles (4) and (5) of my former paper.

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